geometrical transformations

Discovering geometrical transformations in the ancient mosaics of Cyprus:

An instructional approach to Grade 6

Eleni Papageorgiou
Cyprus Pedagogical Institute
Republic of Cyprus
<papageorgiou.e@cyearn.pi.ac.cy>

Constantinos Xenofontos
University of Stirling, UK
<constantinos.xenofontos@stir.ac.uk>

The authors present an exploration of teaching geometrical transformations in a cultural context, using images of ancient mosaics on floors in Cyprus.

In this paper, we present an exploration of teaching geometrical transformations in a cultural context using images of ancient mosaics on floors in Cyprus. Our aim was two-fold: firstly, we intended to help students recognise and distinguish between different types of transformations which appear in mosaics, and secondly, we wanted them to be able to apply these transformations to designing their own mosaics. As instructional tools, the cultural context of these ancient mosaics seemed to motivate students in the exploration of geometrical concepts and properties, while the designing of their own mosaics contributed to a better understanding of various geometrical transformations.

Introduction

Both mathematics and the visual arts constitute fundamental components of the formation of remarkable masterpieces in sculpture, architecture, ceramics and drawing, throughout the history of civilisations from antiquity to today. Aesthetics, harmony, patterns, mathematical precision, and geometrical reasoning co-exist, creating artistic and decorative artefacts, ceremonial artefacts and religious symbols, which reflect the uniqueness of each civilisation. Since ancient Greek and Roman times, mosaics are typical examples of art pieces in which geometry was used to support artistic expression. Geometrical pictures of mythological symbols, which sometimes create mazes and decorative stripes, reveal the application of important mathematical concepts, including geometrical properties and isometric relations (Rousseau, 1999). Even though ancient mosaics are generally perceived as having a decorative role and depict a cultural or social event as a core characteristic, they can be utilised as tools in mathematics teaching and learning (Knuchel, 2004) for facilitating a deeper understanding of many geometrical concepts (Swoboda & Vighi, 2016). In a pedagogical environment of this kind, the simultaneous examination of geometrical concepts and their functionality in an artistic composition enables and strengthens the development of geometrical thinking (Marchis, 2009; Vigni, 2015). The use of ancient mosaics in mathematics lessons sheds light on the interdisciplinary relation of mathematics, the visual arts, and history, by bringing together topics like geometry learning, the history of geometry, and the geometrical dimensions of the arts and of drawing (Karssenberg, 2014). Also, their utilisation in instruction can make associations between the geographical and religious background of these mosaics with mathematical structures (Karssenberg, 2014; Pumfrey & Beardon, 2002) and can contribute to the improvement of learners' cultural and intercultural awareness. Mosaics provide opportunities for students to recognise the role of mathematics in different societies, to acknowledge that mathematical practices have emerged from real-life needs and interests of humans in different regions and eras, and to appreciate the human aspect of mathematics (Zaslavsky, 1999).

The importance of utilising ancient mosaics in mathematics teaching relies on their potential to create a 'cognitive game' and an (inter)active and creative learning environment, as they can promote an investigation of the world of geometry and geometrical structures, while at the same time facilitate the management of visual pieces of information to create new compositions. In particular, mosaics constitute visual models of transformation geometry, whereas analysing and composing them, in attempts to discover patterns, offers opportunities for the development of visualisation skills and geometrical reasoning (Edwards, 1997). Young students can act in the world of patterns by discovering them, on the one hand (Swoboda & Vighi, 2016) and, on the other, reproducing them via an identification process and understanding their underlying structures (Marchini, 2004). Therefore, the act of designing mosaics contributes to an understanding of geometrical transformations and to developing an ability to predict the result of a transformation (Swoboda & Vighi, 2016). It also promotes a deeper understanding of complex concepts that require an understanding of the common structures of algebra and geometry (Bansilal & Naidoo, 2012). Furthemore, it enables students to use abstract mathematical thinking as a tool for their artistic creations (Grzegorczyk & Stylianou, 2006) and allows them to examine and use visual strategies for more effective problem solving (Bansilal & Naidoo, 2012). For the reasons mentioned above, many mathematics textbooks around the world include tasks and ideas for activities of designing patterns and mosaics that require changing the position of a shape with the use of translation, rotation and reflection (Jones & Mooney, 2003). These three types of transformation fall under the linear transformation of isometry, the main characteristic being the preservation of length (Coxeter & Greitzer 1967).

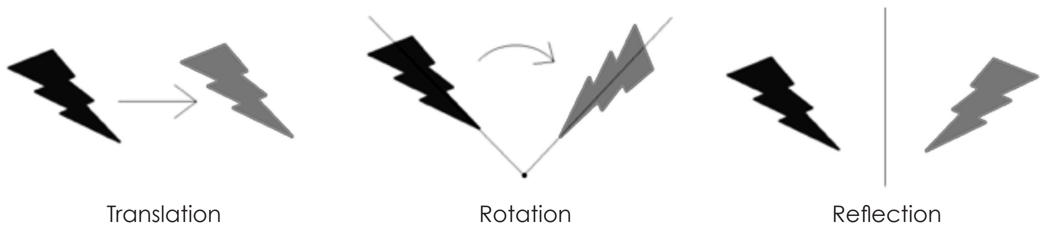

Figure 1: Examples of the geometrical transformations of translation, rotation, and reflection.

Our study was concerned with the designing and implementation of an exploratory approach to teaching geometrical transformations in an applied, real-world context. This approach brings history and the arts in the mathematics classroom and constitutes an alternative, yet complementary suggestion to the purely mathematical context in which the respective concepts are presented in the *National Textbooks of the Republic of Cyprus* (Grade 6, age 11, Part B, pages 94–106) and fits within the *Australian Curriculum: Mathematics*. In the approach proposed, we utilise ancient mosaics from floors in Cyprus, to focus on analysing their patterns and to examine the types of geometrical transformations used by students themselves in designing their own mosaics.

Teaching tools

For this interventional approach, we chose images of the ancient mosaic floors that were discovered in Pafos, Cyprus, in luxurious houses of Roman nobles dating back to the interval between the 2nd and the 5th century AD. Combining aesthetics with mathematics, these mosaics represent some of the finest artistic expressions of Hellenistic and Roman art in Cyprus, and are part of the World Heritage by UNESCO (see whc.unesco.org/en/list/79). These ornate floors in the houses of Aion, Dionysos, Orpheus, and Thiseus (Figure 2) depict scenes from ancient Greek mythology and combine the geometrical transformations of translation, rotation, and reflection on the plane (Natsoulas, 2000).

Figure 2. Photographs of the ancient mosaics of Cyprus used in our intervention. (Photographs used with permission of Pafos Regional Board of Tourism www.visitpafos.org.cy)

Teaching approach

Our approach took place in two stages. The first stage (the stage of exploration) included: (a) a historical description of the ancient floor mosaics and an analysis of the patterns, so that students could recognise the unit-shape (the unit that is repeated in the pattern), and (b) an identification of the various types of transformations that occur on the unit-shape in the creation of the composition. Specifically, students, working in pairs, were asked to describe the mosaic floors. They were also encouraged to identify any mathematical concepts found in the mosaics by using accurate mathematical terminology wherever possible. At the same time, they were reminded of the concepts of 'line of symmetry' and 'symmetric shape' (previously acquired knowledge).

Below, an extract from the discussion between a student (Maria), and the teacher is presented.

Maria: This floor (Figure 3) has square tiles put next to each other without any space between them. Every square tile has different repeating smaller shapes, like small and big rhombuses. (Note: see first square on top left side of the picture.)

Teacher: In which ways are these small and big rhombuses repeating patterns?

Maria: I imagine two axes of symmetry, a horizontal and a vertical one, both in the middle of the square. On the left side of the horizontal axis there is a 'horizontal' rhombus, half of it above the axis and the other below, and it ends in the middle of the square. The same shape begins where the first rhombus finishes, only on the right side of the square, in the same way that we have the two 'vertical' rhombuses on the vertical axis.

Figure 3. Detail from Figure 2.

Teacher: Suppose we only have one rhombus which moves each time to create a whole picture. How would you describe the change of its position each time? Let's say, this one (note: the teacher points at the big rhombus on the top part of the square) is the only rhombus we have in the beginning.

Maria: Hmm. If it flips below, then we have the other red rhombus.

Teacher: Good! What about the black rhombuses? How can we get these?

Maria: Hmm. That's hard (stops for a while and thinks). Oh wait, if we turn the red one half a circle, it will go there!

Teacher: Good idea! But half a circle? Think again. Where will this go if it turns half a circle?

Maria: Oh, right. It will go to the red rhombus below. Wait, maybe one quarter of the circle?

Teacher: One quarter of the circle! Good job!

Like the student's description above, most descriptions implied an intuitive recognition of various transformations. With the help of the classroom teacher and two researcher-authors, students named them as translation, rotation around a point, and/or reflection.

The second stage (the stage of creation/designing) focused on children's designing of mosaics, by applying different types of transformations on geometrical shapes or shape compositions. At this stage, students took the role of mosaic designers and created their own compositions, by using various geometrical skills that were previously acquired, like drawing equilateral triangles, pentagons, and hexagons (Hogendijk, 2012). To be specific, the second stage focused on:

(a) students' descriptions of their own mosaics with the use of accurate mathematical terminology, like "translation x units to right/left/up/down" (in relation to imaginary horizontal and vertical axes that intersect at the left down edge of each shape), "rotation of x degrees, from the right to the left" or "from the left to the right" (in relation to an imaginary point), "reflection" (in relation to an imaginary horizontal, vertical, or diagonal axis); and,

(b) grouping students' mosaics according to types and combinations of geometrical transformations included in them.

Accuracy in describing mosaics aims at a deeper understanding of concepts, while grouping outcomes according to the types of transformations employed promotes, even intuitively, the unity of mathematical groups (Natsoulas, 2000).

The intervention was applied in a classroom of 22 students (10 boys and 12 girls) at Grade 6 (age 11) of an urban elementary school in the Republic of Cyprus. The implementation took place in two 80-minute teaching periods over two consecutive days. The first teaching period was concerned with the first stage (the stage of exploration), and the second period was used for the second stage (the stage of creation/designing). Teaching was led by the classroom teacher, while the two authors of the paper had the role of participant/observer: they would ask students questions during the lessons, and, when needed, they would encourage them to observe, further analyse, and check the information at hand. Furthermore, the researchers took notes on the difficulties some students faced their cognitive behaviour and explanations. During the second stage, the researchers invited students to identify the geometrical transformations they had used and to describe them in accurate mathematical terminology.

A look at students' designs

The whole interventional approach intrigued students and aroused their curiosity to investigate the mathematical concepts in the ancient mosaics. They called this particular mathematics lesson "amusing", "different", "pleasant", and "a game of discovery and designing". They also expressed positive opinions regarding the importance of geometry in design.

In students' mosaics, we can see one to three types of geometrical transformations. Figure 4 shows some representative drawings from the class. In drawings 'a' and 'b', the students used only translation of the unit-shape, both horizontally and vertically on imaginary axes. In drawing 'a', each of the distinct shapes was translated horizontally to the left and then vertically up (heart) or down (rectangle), creating a simple composition. In drawing 'b', at first a unit-shape was designed, followed by its horizontal translation. The unity of the two (initial shape and its translation) created a new unit-shape, and its horizontal translation created a stripe. Subsequently, the stripe was translated vertically.

Let's have a look at Yiannis' description of his own drawing (drawing 'b'):

> In my mosaic, I drew a horizontal axis with white and blue squares. First, I drew a blue square. The next one remained white. Then, I repeated the pattern blue-white-blue-white, to make a horizontal axis. In other words, I translated the blue square two units to the right. Because I wanted my axis to have two stripes of blue and white squares, I translated the whole stripe one unit below and then one unit to the right. After, I made a drawing in black and white colour, above the blue-white axis, on the left side of the paper. I left two empty columns and then repeated the black-green shape. I then translated the pair of two shapes five units to the left. Finally, I translated the whole shape two units below the white axis.

In drawings 'c' and 'd', the compositions emerged from the reflection of a unit-shape vertically or horizontally. In drawing 'e', we see a combination of reflection and translation. Nevertheless, in these three drawings, the mosaics are created with the use of an imaginary Cartesian system of axes with its origin at the centre of the paper. Below, we present Sofia's description of her drawing (drawing 'e'):

> I drew something like a capital L in English while looking from the other side. Then I drew its reflection over an imaginary vertical axis, and coloured it blue. After, I reflected the whole shape over an imaginary horizontal axis, and coloured it green and red. Finally, I reflected the whole shape over a new vertical axis passing from the middle of the paper.

As we can see from all drawings, translation and reflection are the most commonly used types of transformation. Rotation around a point was applied only in three drawings, 'i' (in the design of the triangle), 'h' (as can be observed in the tongue and nose of the face), and 'l'.

In regards to spatial awareness, in drawings 'a–e' the perception of space as being limited to well-defined boundaries is observed. This is implied by students' 'frames' around their drawings (i.e., drawings 'c' and 'd') (Swoboda & Vighi, 2016) or by the fact that their drawing is completed within an imaginary frame (i.e,. drawings 'a', 'b', and 'e'). Other students, as we can see in drawings 'f–m', seem to perceive space as unlimited, as the repeated reflections or translations of the unit-shape allude to a subconscious perception of infinity (Marchini, 2004). Even though in drawings 'f–i' the pattern is restricted within the limits of the piece of paper, this was probably a decision to preserve the drawing's regularity and for aesthetic reasons (i.e. to have a complete shape that isn't cut). Finally, in drawings 'c–k', students appear to perceive the space as anisotropic, that is, "there are two privileged directions, horizontal and vertical", while, in drawings 'l' and 'm', space is perceived as isotropic, that is, "without distinction among directions" (Swoboda & Vighi, 2016, p. 32).

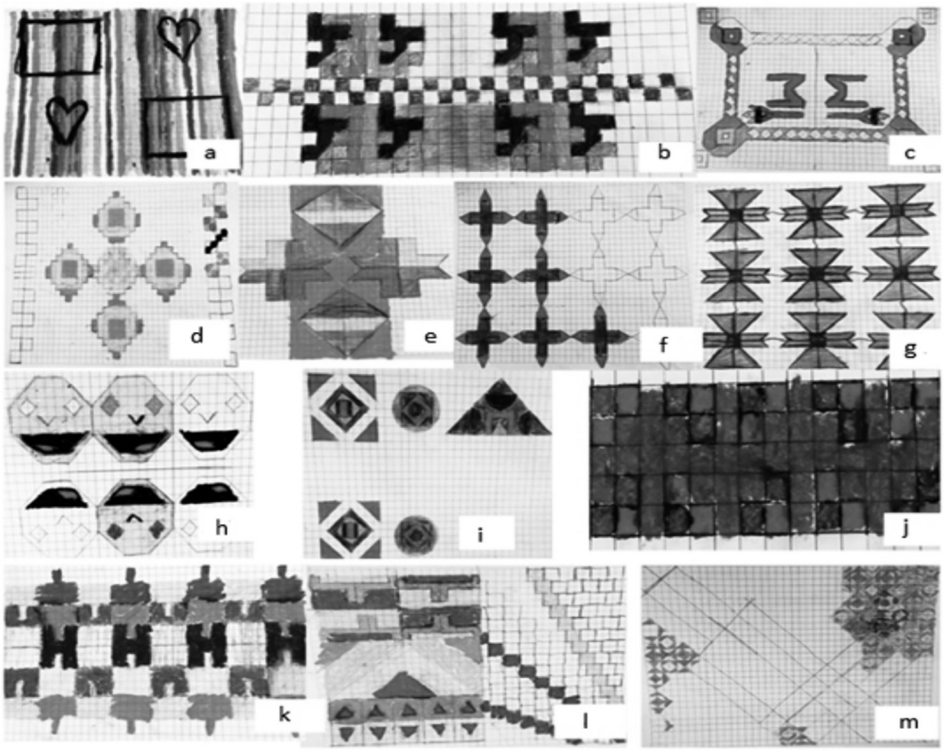

Figure 4. Students' mosaics.

In the lesson that followed our intervention, the teacher asked students to work on the corresponding tasks from the National Textbooks in mathematics, as practice. While we didn't have the opportunity to observe that lesson, our a-posteriori conversation with the teacher revealed that the majority of students understood the concepts and were able to apply their emerged knowledge to tasks of purely mathematical contexts, like those that appear in the textbooks.

Conclusions

In this intervention, we designed and implemented an exploratory instructional approach around the teaching of geometrical transformations, by utilising the ancient mosaics of Cyprus as instructional tools. Bringing together visual arts, history, and mathematics, the proposed intervention led students to examine the geometrical transformations of shapes, their functional arrangement in space and in the real world, and to appreciate the contribution of mathematics in the history of civilisations. Students examined the ancient mosaics though a mathematical lens (Karssenberg, 2014; Vighi, 2015), and described the ancient mosaics in accurate mathematical language and geometrical transformations (Swoboda & Vighi, 2016). However, the design of mosaics by students showed that translation and reflection were more commonly employed than rotation around a point. Interestingly, this finding is not in accordance with Hollebrands' (2004) study with high-school students in the USA, in which students had more difficulties in understanding translation. To conclude, this intervention, with the utilisation of the ancient mosaics, increased students' interest and participation (as commented by the classroom teacher) and challenged them to explore the concepts, and promoted the use of skills like observation, grouping, arrangement, analysis, composition, and in general, spatial awareness, directions and isometry (Marchini & Vighi, 2011).

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